19. Mar, 2024

Q1. Show that $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$$

Solution:

Suppose that $0 < |z| < 4$, then $0 < \frac{|z|}{4} < 1$. We apply the expansion $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$ to derive

$$\frac{1}{4z - z^2} = \frac{1}{4z(1 - z/4)} = \frac{1}{4z} \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n = \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$$

$$= \frac{1}{4z} + \sum_{n=1}^{\infty} \frac{z^{n-1}}{4^{n-1}}$$

replacing n by $n+1$

$$= \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+1}}$$

Q2. Let $f(z)$ denote a function which is analytic in some annular domain about the origin that includes the unit circle $z = e^{i\phi}$, $-\pi \le \phi \le \pi$.

(a) Show that

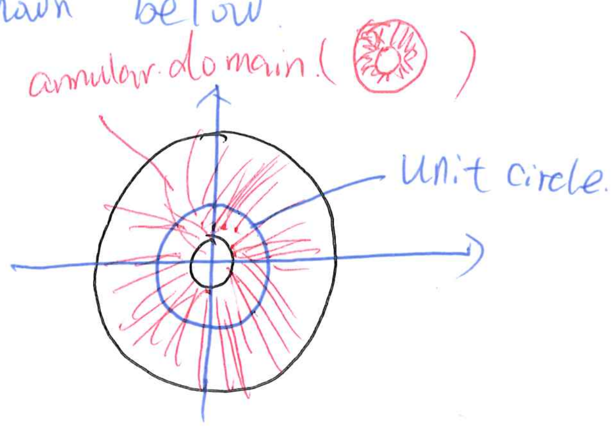
$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\,d\phi + \frac{1}{2\pi}\sum_{n\ge1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right] d\phi.$$

(b) Write $u(\theta) = \mathrm{Re}\,[f(e^{i\theta})]$ and show how it follows from the expansion in part (a) that

$$u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)\,d\phi + \frac{1}{\pi}\int_{-\pi}^{\pi} u(\phi)\cos[n(\theta-\phi)]\,d\phi.$$

Solution:

The function $f(z)$ is analytic in some domain centered at origin and the unit circle $C: z = e^{i\phi}$, $-\pi \le \phi \le \pi$ is contained in that domain, as shown below.



(a) For each z in the annular domain, there is a Larent sereries representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n\geq 1}^{\infty} b_n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_C \frac{f(t)}{t^{n+1}} dt \right) z^n + \sum_{n\geq 1}^{\infty} \left(\frac{1}{2\pi i} \int_C \frac{f(t)}{t^{-n+1}} dt \right) z^{-n}$$

where c is the unit circle. We replace t by $e^{i\phi}$, $-\pi \leq \phi \leq \pi$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(e^{i\phi})}{(e^{i\phi})^{n+1}} i e^{i\phi} d\phi z^n + \sum_{n\geq 1}^{\infty} \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(e^{i\phi})}{(e^{i\phi})^{-n+1}} i e^{i\phi} d\phi z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) \left(\frac{z}{e^{i\phi}}\right)^n d\phi + \sum_{n\geq 1}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) \left(\frac{e^{i\phi}}{z}\right)^n d\phi$$

$$= \frac{1}{2\pi} \left\{ \int_{-\pi}^{\pi} f(e^{i\phi}) d\phi + \sum_{n\geq 1}^{\infty} \int_{-\pi}^{\pi} f(e^{i\phi}) \left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n \right] d\phi \right\}$$

(b). Putting $z = e^{i\theta}$ in the final result of (a) and using $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$$f(e^{i\theta}) = \frac{1}{2\pi} \left\{ \int_{-\pi}^{\pi} f(e^{i\phi}) d\phi + \sum_{n\geq 1}^{\infty} \int_{-\pi}^{\pi} f(e^{i\phi}) \left[\left(\frac{e^{i\theta}}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{e^{i\theta}}\right)^n \right] d\phi \right\}$$

$$= \frac{1}{2\pi} \left\{ \int_{-\pi}^{\pi} f(e^{i\phi}) d\phi + \sum_{n\geq 1}^{\infty} \int_{-\pi}^{\pi} f(e^{i\phi}) 2\cos[n(\theta-\phi)] d\phi \right\}$$

Equating the real part on each side yields.

$$u(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\phi) d\phi + \sum_{n=1}^{\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} u(\phi) \cos[n(\theta-\phi)] d\phi.$$

3. Prove that if

$$f(z) = \begin{cases} \dfrac{e^z - 1}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0. \end{cases}$$

then f is an entire function.

Solution:

when $z \neq 0$, $f(z)$ has the power series representation

$$f(z) = \frac{1}{z}\left[\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^k}{k!} + \cdots\right) - 1\right]$$

$$= 1 + \frac{z}{2!} + \cdots + \frac{z^{k-1}}{k!}$$

Since this representation clearly holds for $z=0$ too, it is actually valid for all z. Hence f is entire.

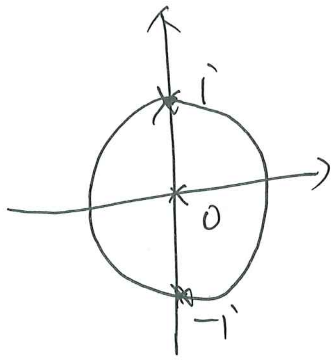
4. Use multiplication of series to show that.

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \cdots \qquad (0 < |z| < 1)$$

Solution:

The singularities of $f(z) = \dfrac{e^z}{z(z^2+1)}$ are $z = 0, \pm i$.

The problem here is to find the Laurent series for f that is valid in the punctured disk $0 < |z| < 1$, shown below



Recall that
$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^k}{k!} + \cdots, \qquad |z| < \infty$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad |z| < 1.$$

which enable us to write

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 - \cdots \qquad |z| < 1.$$

and
$$\begin{aligned}\frac{e^z}{1+z^2} = 1 + z + \frac{z^2}{2} &+ \frac{z^3}{6} + \frac{z^4}{24} + \cdots \\ - z^2 &- z^3 - \frac{z^4}{2} - \cdots \\ &+ z^4 + z^5 + \frac{z^6}{2!} + \cdots\end{aligned}$$

$$= 1 + z - \frac{z^2}{2} - \frac{5}{6}z^3 + \frac{13}{24}z^4 + \cdots$$

5. Let $f(z)$ be an entire function that is reprensented by a series of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad (|z| < \infty)$$

(a) By differentiating the composite function $g(z) = f[f(z)]$ successively, find the first three nonzero terms in the Taylor series for $g(z)$ and show that

$$f[f(z)] = z + 2a_2 z^2 + 2(a_2^2 + a_3) z^3 + \cdots \qquad (|z| < \infty)$$

(b) By applying the resule in part (a) to the function $f(z) = \sin z$, show that $\sin(\sin z) = z - \frac{1}{3} z^3 + \cdots \qquad |z| < \infty$

Solution

(a) $$g(z) = f[f(z)] = g(0) + \frac{g'(0)}{1!} z + \frac{g''(0)}{2!} z^2 + \frac{g'''(0)}{3!} z^3 + \cdots$$

It is straightforword to show that

$$g'(z) = f'[f(z)]\, f'(z)$$

$$g'(0) = f'[f(0)]\, f'(0) = f'(0)\, f'(0) = 1$$

Since $f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots$

$f'(0) = 1 \quad \Rightarrow g'(0) = 1$

$$g''(z) = f''[f(z)]\,(f'(z))^2 + f'[f(z)]\,f''(z)$$

$$g''(0) = f''[f(0)]\,(f'(0))^2 + f'[f(0)]\,f''(0)$$

$$= f''(0)\cdot 1 + 1\cdot f''(0)$$

Since $f''(z) = 2a_2 + 6a_3 z + 12a_4 z^2 + \cdots$

$$f''(0) = 2a_2 \quad \Rightarrow \quad g''(0) = 4a_2$$

$$g'''(z) = f'''[f(z)]\,[f'(z)]^3 + f''[f(z)]\,2f'(z)\cdot f''(z)$$

$$+ f''[f(z)]\,f'(z)\,f''(z) + f'[f(z)]\,f'''(z)$$

$$g'''(0) = f'''(0)\cdot 1 + 2f''(0)\,f'(0)\,f''(0) + f''(0)\,f'(0)\,f''(0)$$

$$+ f'(0)\,f'''(0)$$

Since $f'''(z) = 6a_3 + 24a_4 z + \cdots$

$$f'''(0) = 6a_3 \quad \Rightarrow \quad g'''(0) = 6a_3 + 8a_2^2 + 4a_2^2 + 6a_3$$

Hence

$$= 12(2a_2^2 + a_3)$$

$$f[f(z)] = z + \frac{4a_2}{2!}z^2 + \frac{12(2a_2^2+a_3)}{3!}z^3 + \cdots$$

$$= z + 2a_2 z^2 + 2(2a_2^2 + a_3)z^3 + \cdots$$

(b) Since $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \quad |z| < \infty$

Using the result in part (a) with $a_2 = 0$, $a_3 = -\frac{1}{3!} = -\frac{1}{6}$, then

$\sin(\sin z) = z - \frac{1}{3} z^3 + \cdots$